5. Duboshin, G. N., Celestial Mechanics. Basic Problems and Methods. "Nauka" Moscow, 1968.
6. Erike, K., Space Flight. Vol. 1, Fizmatgiz, Moscow, 1963.

Translated by M. D. V. D.

ON DISSIPATIVE BEHAVIOR OF AN EQUATION OF NONLINEAR OSCILLATIONS

PMM Vol. 35, №5, 1971, pp. 933-937
Iu. N. BIBIKOV and K. D. SANOVSKII
(Leningrad)
(Received July 2, 1970)

Sufficient conditions are obtained for dissipative behavior of the following equation

$$x^{\cdot\cdot} + f(x^{\cdot},\ x) + g(x) = e(t) \tag{1}$$

This equation is called dissipative [1] if for any of its solutions the functions $x(t)$ and $x^{\cdot}(t)$ are uniformly finally bounded for $t \to \infty$. The conditions found here differ from those already known [1, 2] because in the conditions here the functions f and g can be bounded and arbitrarily small in comparison to the force term $e(t)$. Namely, the following theorem is valid.

Theorem. Let the following conditions be satisfied.

1) Piecewise continuous functions $f(z,\ x)$, $g(x)$ and $e(t)$ are defined for all values of $x, z \in (-\infty, \infty)$ and $t \in [0, \infty)$. These functions ensure the existence of a solution of equation (1) in any point of the phase plane $xx^{\cdot}$ for any $t \geqslant 0$.

2) $e(t) = e_1(t) + e_2(t), \qquad |e_1(t)| \leqslant e_{10} < \infty$

$$E_2(t) = \int_0^t e_2(t)\,dt, \qquad |E_2(t)| \leqslant E_{20} < \infty$$

3) Nondecreasing piecewise continuous functions φ and ψ exist such that

$$\varphi(z) \leqslant f(z,\ x) \leqslant \psi(z), \qquad x,\ z \in (-\infty,\ \infty),\ z\varphi(z) \geqslant 0, \qquad z\psi(z) \geqslant 0$$
$$e_{10} < \sup \varphi(z), \quad -e_{10} > \inf \psi(z)$$

4) $xg(x) \geqslant 0$

$$\varliminf_{x\to\infty} g(x) - e_{10} > -\varphi(-2E_{20} - 0) \geqslant 0, \qquad \varlimsup_{x\to-\infty} g(x) + e_{10} < -\psi(2E_{20} + 0) \leqslant 0$$

5) $|g(x)| \leqslant g_0 < \infty$

Then equation (1) is dissipative.

Proof 1. Instead of Eq. (1) let us examine the equivalent system

$$\dot{x} = y + E_2(t), \qquad y^{\cdot} = -f(y + E_2(t), x) - g(x) + e_1(t) \tag{2}$$

Let

$$\left.\frac{dy}{dx}\right|_{(2)} = \frac{-f(y + E_2, x) - g(x) + e_1(t)}{y + E_2}$$

In the xy plane let us investigate the curves

$$\Gamma(H, \alpha) = \{(x, y) : {}^1/_2 y^2 + G(x) + \alpha x = H = \text{const}\}, \qquad G(x) = \int_0^x g(x)\,dx$$
$$-\infty < \alpha, H < \infty$$

Along these curves

$$\left.\frac{dy}{dx}\right|_{\Gamma} = -\frac{g(x)+\alpha}{y}$$

By virtue of Conditions (3) and (5) of the theorem a quantity $b > E_{20}$ exists such that

$$y\,[\varphi(y - E_{20}) - e_{10}] - g_0 E_{20} > 0 \qquad (y \geqslant b)$$

$$y\,[\psi(y + E_{20}) + e_{10}] - g_0 E_{20} > 0 \qquad (y \leqslant -b)$$

We obtain α_0 from conditions

$$0 < \alpha_0 \leqslant \frac{b\,[\varphi(b - E_{20}) - e_{10}] - g_0 E_{20}}{b + E_{20}}$$

$$0 < \alpha_0 \leqslant \frac{-b\,[\psi(-b + E_{20}) + e_{10}] - g_0 E_{20}}{b + E_{20}}$$

Then we can verify by a direct check that

$$\left.\frac{dy}{dx}\right|_{\Gamma(H,\alpha_0)} \geqslant \left.\frac{dy}{dx}\right|_{(2)} \quad (y \geqslant b), \quad \left.\frac{dy}{dx}\right|_{\Gamma(H,-\alpha_0)} \geqslant \left.\frac{dy}{dx}\right|_{(2)} \quad (y \leqslant -b) \tag{3}$$

2. From Condition (4) of the theorem follows that $G(x)$ does not decrease for $x \geqslant 0$ and does not increase for $x \leqslant 0$. Furthermore,

$$G(x) \geqslant 0, \quad G(0) = 0, \quad \lim G(x) = \infty \quad \text{for } x \to \pm\infty \tag{4}$$

Let us form the functions

$$m(C) = \max\{x\colon G(x) = C\}, \qquad n(C) = \min\{x\colon G(x) = C\}, \qquad C \geqslant 0$$

$$\mu(C) = \max\{x\colon G(x) + \alpha_0 x = C\}, \qquad \inf\{G(x) + \alpha_0 x\} \leqslant C < \infty$$

$$\nu(C) = \min\{x\colon G(x) - \alpha_0 x = C\}, \qquad \inf\{G(x) - \alpha_0 x\} \leqslant C < \infty$$

By virtue of (4), $m(C)$ and $\mu(C)$ are strictly increasing; $n(C)$ and $\nu(C)$ are strictly decreasing functions

$$m(C) \geqslant 0, \quad n(C) \leqslant 0, \quad \mu(0) = \nu(0) = 0$$

$$m(C) \to \infty, \quad n(C) \to -\infty, \quad \mu(C) \to \infty, \quad \nu(C) \to -\infty \quad \text{for } C \to \infty$$

Let us examine the differences $m(C) - \mu(C)$ and $\nu(C) - n(C)$ for $C \geqslant 0$. Using the equality

$$G(\mu(C)) + \alpha_0 \mu(C) = C = G(m(C))$$

and Condition (5) of the theorem, we find

$$\alpha_0 \mu(C) = G(m) - G(\mu) \leqslant g_0 (m - \mu)$$

From here it follows that

$$m(C) - \mu(C) \to \infty \quad \text{for } C \to \infty \tag{5}$$

In an analogous manner we also prove the following relationship

$$\nu(C) - n(C) \to \infty \quad \text{for } C \to \infty \tag{6}$$

3. Setting $\varkappa(x', x''; \alpha) = \max\{G(\xi) + \alpha\xi\}$, $\xi \in [x', x'']$, we examine the following functions for $C \geqslant 0$:

$$\sigma(x; C) = \sqrt{2}\,\sqrt{{}^1\!/_2\, b^2 + \varkappa(n, x; \alpha_0) - G(x) - \alpha_0 x}\,, \quad x \geqslant n(C)$$

$$\tau(x; C) = -\sqrt{2}\,\sqrt{{}^1\!/_2\, b^2 + \varkappa(x, m; -\alpha_0) - G(x) + \alpha_0 x}\,, \quad x \leqslant m(C)$$

The inequalities below are obvious

$$\sigma(x;\ C) \geqslant b, \qquad \tau(x;\ C) \leqslant -b$$

Since $G(x) + \alpha x$ is continuous in x, then $\varkappa(x', x''; \alpha)$ is continuous in x' and x''. Therefore functions σ and τ are continuous in x.

Let now
$$k(C) = \min\{x' : x \geqslant x' \geqslant n(C) \Rightarrow \varkappa(n, x; \alpha_0) = G(x) + \alpha_0 x\}$$
$$l(C) = \max\{x' : x \leqslant x' \leqslant m(C) \Rightarrow \varkappa(x, m; -\alpha_0) = G(x) - \alpha_0 x\}$$
Functions $k(C)$ and $l(C)$ are characterized by
$$k(C) = \min\{x' : x \geqslant x' \geqslant n(C) \Rightarrow \sigma(x; C) = b\}$$
$$l(C) = \max\{x' : x \leqslant x' \leqslant m(C) \Rightarrow \tau(x; C) = -b\}$$
From the determination of $k(C)$ and $l(C)$ we have directly
$$G(k) + \alpha_0 k = \max_{\xi \in [n, k]}\{G(\xi) + \alpha_0\xi\} \leqslant \max_{\xi \in [n, 0]}\{G(\xi) + \alpha_0\xi\} \leqslant G(n) = C$$
$$G(l) - \alpha_0 l = \max_{\xi \in [l m]}\{G(\xi) - \alpha_0\xi\} \leqslant \max_{\xi \in [0, m]}\{G(\xi) - \alpha_0\xi\} \leqslant G(m) = C$$
From here it follows that
$$k(C) \leqslant \mu(G(k) + \alpha_0 k) \leqslant \mu(C), \qquad l(C) \geqslant \nu(G(l) - \alpha_0 l) \geqslant \nu(C)$$
Then, utilizing (5) and (6), we arrive at the statement
$$m(C) - k(C) \to \infty, \quad l(C) - n(C) \to \infty \text{ for } C \to \infty \tag{7}$$

4. In the xy plane let us examine the location of trajectories of system (2) with respect to curves
$$\Sigma(C) = \{(x, y) : x \geqslant n(C),\ y = \sigma(x; C)\}$$
$$T(C) = \{(x, y) : x \leqslant m(C),\ y = \tau(x; C)\}$$

Functions $\varkappa(n, x; \alpha_0)$ and $\varkappa(x, m; -\alpha_0)$ have piecewise continuous derivatives. Therefore
$$\left.\frac{dy}{dx}\right|_\Sigma = \frac{d\sigma}{dx} = \begin{cases} -(g(x) + \alpha_0)/y & \text{for } d\varkappa(n, x; \alpha_0)/dx = 0 \\ 0 & \text{for } d\varkappa(n, x; \alpha_0)/dx = g(x) + \alpha_0 > 0 \end{cases}$$
$$\left.\frac{dy}{dx}\right|_T = \frac{d\tau}{dx} = \begin{cases} -(g(x) - \alpha_0)/y & \text{for } d\varkappa(x, m; -\alpha_0)/dx = 0 \\ 0 & \text{for } d\varkappa(x, m; -\alpha_0)/dx = g(x) - \alpha_0 < 0 \end{cases}$$

Then, taking into account (3) along curves Σ and T, we obtain the following inequalities:
$$\left.\frac{dy}{dx}\right|_\Sigma \geqslant -\frac{g(x) + \alpha_0}{y} = \left.\frac{dy}{dx}\right|_{\Gamma(^1/_2 y^2 + G(x) + \alpha_0 x;\, \alpha_0)} \geqslant \left.\frac{dy}{dx}\right|_{(2)}$$
$$\left.\frac{dy}{dx}\right|_T \geqslant -\frac{g(x) - \alpha_0}{y} = \left.\frac{dy}{dx}\right|_{\Gamma(^1/_2 y^2 + G(x) - \alpha_0 x;\, -\alpha_0)} \geqslant \left.\frac{dy}{dx}\right|_{(2)} \tag{8}$$

5. According to Condition (4) of the theorem a quantity $a > 0$ exists such that
$$B = \inf_{|x| \geqslant a}\{|g(x)| - e_{10}\} > 0, \quad \varphi(-2E_{20} - 0) > -B, \quad \psi(2E_{20} + 0) < B \tag{9}$$

Let us examine in the region $x \geqslant a$, $-E_{20} \leqslant y \leqslant b$ the curves $\Phi(x_0)$, which are determined by the differential system
$$x^{\cdot} = y + E_{20}, \quad y^{\cdot} = -\varphi(y - E_{20}) - B; \qquad t = t_0 \quad x = x_0, \quad y = b$$

Integrating this system, we find that
$$\Phi(x_0) = \left\{(x, y) : x - x_0 = \int_{y - E_{20}}^{b - E_{20}} \frac{z + 2E_{20}}{\varphi(z) + B}\, dz\right\}$$
From (9) and the Condition (3) of the theorem it is evident that along the curves $\Phi(x_0)$ the following inequalities are satisfied:

$$\left.\frac{dx}{dt}\right|_{\Phi} \geqslant 0, \qquad \left.\frac{dx}{dy}\right|_{\Phi} \leqslant 0, \qquad \left.\frac{dx}{dt}\right|_{(2)} \leqslant \left.\frac{dx}{dt}\right|_{\Phi} \tag{10}$$

$$\left.\frac{dy}{dt}\right|_{(2)} \leqslant \left.\frac{dy}{dt}\right|_{\Phi} \leqslant -\varphi(-2E_{20}-0) - B < 0$$

Then

$$\Delta_1 = \max(x - x_0) = \int\limits_{-2E_{20}}^{b-E_{20}} \frac{z + 2E_{20}}{\varphi(z) + B}\, dz$$

In the region $x \leqslant -a$, $-b \leqslant y \leqslant E_{20}$ the following system is examined in an analogous manner

$$x^{\cdot} = y - E_{20}, \quad y^{\cdot} = -\psi(y + E_{20}) + B; \qquad t = t_0 \quad x = x_0, \quad y = -b$$

We obtain the family of curves

$$\Psi(x_0) = \left\{ (x, y) : x_0 - x = \int\limits_{-b+E_{20}}^{y+E_{20}} \frac{-z + 2E_{20}}{-\psi(z) + B}\, dz \right\}$$

along which

$$\left.\frac{dx}{dt}\right|_{\Psi} \leqslant 0, \qquad \left.\frac{dx}{dy}\right|_{\Psi} \leqslant 0, \qquad \left.\frac{dx}{dt}\right|_{(2)} \geqslant \left.\frac{dx}{dt}\right|_{\Psi} \tag{11}$$

$$\left.\frac{dy}{dt}\right|_{(2)} \geqslant \left.\frac{dy}{dt}\right|_{\Psi} \geqslant -\psi(2E_{20}+0) > 0$$

$$\Delta_2 = \max(x_0 - x) = \int\limits_{-b+E_{20}}^{2E_{20}} \frac{-z + 2E_{20}}{-\psi(z) + B}\, dz$$

6. In this subsection, a region will be constructed in the xy plane. All trajectories of system (2) enter into this region. Let

$$C_1 = G(a + \Delta_1), \qquad C_2 = G(-a - \Delta_2)$$

On the basis of (7) we can select such C_3 and C_4, that

$$C_3 = \min\{C' : C \geqslant C' \Rightarrow m(C) - k(C) \geqslant \Delta_1\}$$
$$C_4 = \min\{C' : C \geqslant C' \Rightarrow l(C) - n(C) \geqslant \Delta_2\}$$

Then we assume $C_5 = \max(C_1, C_2, C_3, C_4)$. By virtue of this the following relationships are valid for all $C \geqslant C_5$

$$m(C) - \Delta_1 \geqslant a, \qquad \sigma(m(C) - \Delta_1;\ C) = b$$
$$n(C) + \Delta_2 \leqslant -a, \qquad \tau(n(C) + \Delta_2;\ C) = -b$$

With each value $C \geqslant C_5$ we associate some closed curve $\omega(C)$, which is obtained when the following curves intersect: $\Sigma(C)$, $\Phi(m(C) - \Delta_1)$, $x = m(C)$, $T(C)$, $\Psi(n(C) + \Delta_2)$ and $y = n(C)$, (see Fig. 1.) Closed regions of the xy plane, bounded by curves $\omega(C)$ are designated by $\Omega(C)$. It follows from (9) that for $|x| \geqslant a$ the functions $m(C)$ and $n(C)$ are continuous. Therefore the family of curves $\omega(C)$ is also continuous

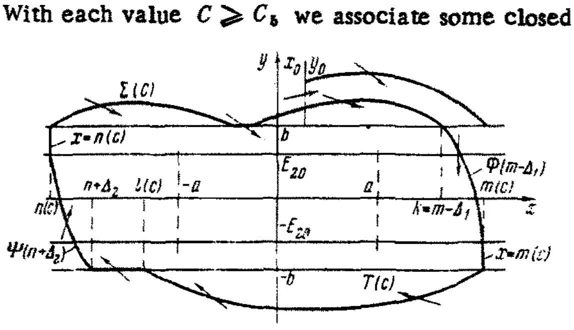

Fig. 1.

and fills the region $\Omega(C^*) \setminus \Omega(C_5)$ for $C^* \geqslant C_5$.

The trajectory, leaving an arbitrary point (x_0, y_0), after a finite segment of time enters into the band $-b \leqslant y \leqslant b$, i.e. it turns out to be in some region $\Omega(C)$. In fact, for the sake of definiteness let $y_0 > b$. Since $G(x) + \alpha_0 x \to \infty$ for $x \to \infty$, the curve $\Gamma(^1/_2 y_0^2 + G(x_0) + \alpha_0 x_0;\ \alpha_0)$ for $x > x_0$ intersects the straight line $y = b$ Then by virtue of (3) and the inequality $b > E_{20}$ the trajectory also intersects this line and enters into the band $-b \leqslant y \leqslant b$.

From inequalities (8), (10), (11) and $|E_2(t)| \leqslant E_{20}$ it follows that the trajectories cross the curves $\omega(C)$ from the outside into the region $\Omega(C)$ for any $C \geqslant C_5$. Consequently, in the region $\Omega(C^*) \setminus \Omega(C_5)$ $(C^* \geqslant C_5)$ the quantity C decreases monotonically along the trajectory. If the existence of the limit $C^+ \geqslant C_5$ is assumed here, then this will indicate that the trajectory winds up from the outside onto the curve $\omega(C^+)$. In particular, in the region $x \geqslant a$, $-E_{20} > y > -b$ the function $x^{\cdot}|_{(2)}$ becomes arbitrarily close to zero. Howevet, this is in contradiction to the first equation of system (2) and the inequality* $|E_2(t)| \leqslant E_{20}$.

Thus, in the course of time all trajectories of system (2) get into the region $\Omega(C_5)$ and subsequently remain in it. This completes the proof of the theorem.

Note: Under the conditions of the theorem the requirement $|g(x)| \leqslant g_0 < \infty$ is essential. Thus, for the equation

$$x^{\cdot\cdot} + \operatorname{sign} x^{\cdot} + x = \sin t$$

the statement of the theorem is not valid [3].

BIBLIOGRAPHY

1. Pliss, V.A., Nonlocal Problems of the Theory of Oscillations. Academic Press, New York and London, 1966.
2. Reissig, R., Sansone, G., and Conti, R., Qualitative Theorie nichtlinearer Diefferentialgleichungen. Roma, Edizioni Cremoneze, 1963.
3. Iorish, Iu.I., Vibration Measurements. M., Mashgiz, 1956.

Translated by B. D.

BIFURCATIONS IN THE VICINITY OF A "FUSED FOCUS"

PMM Vol. 35, №5, 1971, pp. 937-941
N. A. GUBAR'
(Gor'kii)
(Received November 26, 1970)

Conditions are presented for the existence of bifurcation of a singular point of the type of a "fused focus". The fusing is accomplished with ordinary trajectories under the assumption that the general integrals are known for both systems forming the "fused system".

In the approximation of analytical characteristics in the equations of motion of dynamic systems by piecewise linear or relay functions on the lines of fusing, singular points can arise which are fused from ordinary or singular trajectories of systems to be fused. When the parameters of the system change, analogies